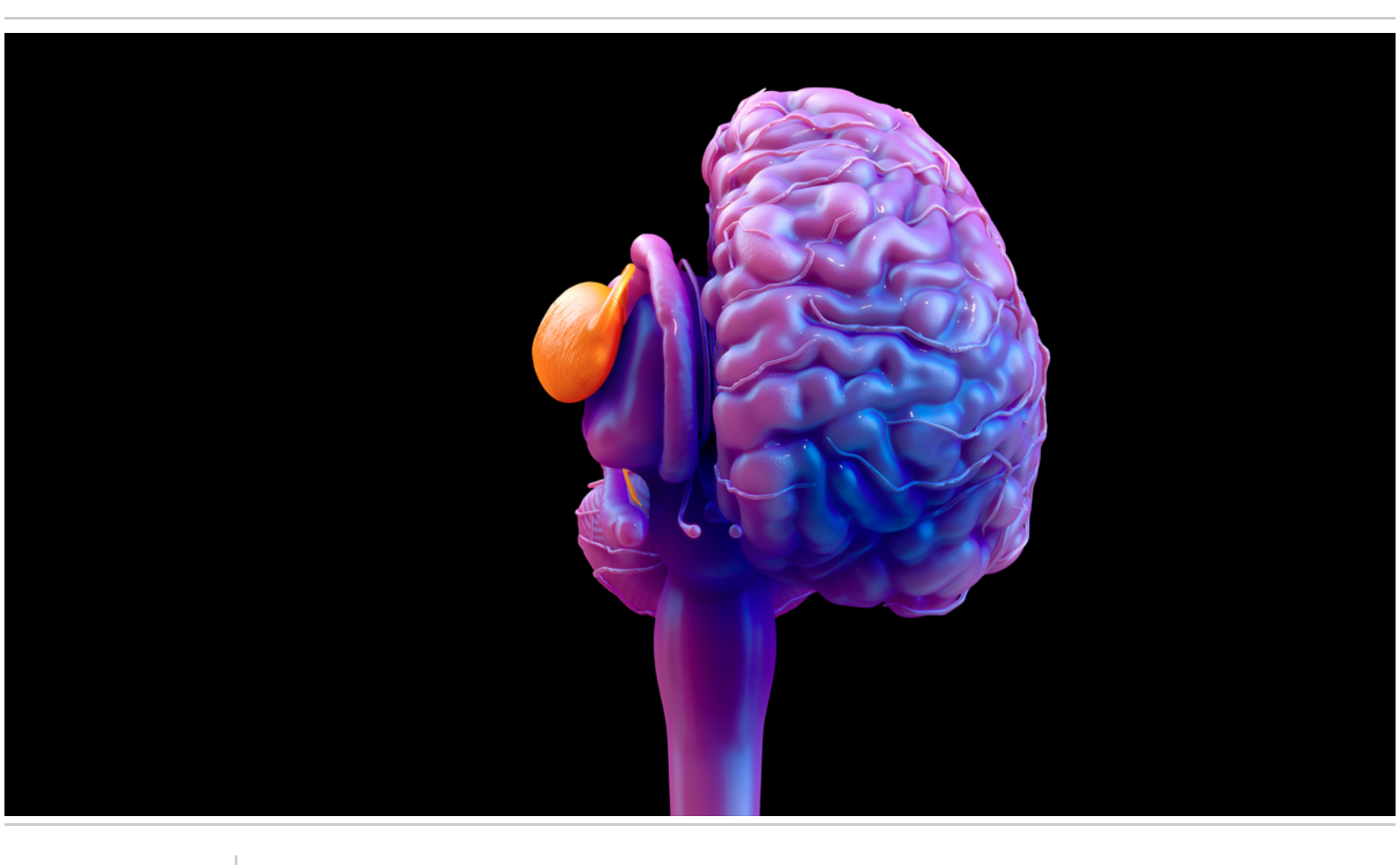
Cosmos » Magazine

The art of logical thinking



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COSMOS
Cosmos

Cosmos is a quarterly science magazine. We aim to inspire curiosity in 'The Science of Everything' and make the world of science accessible to everyone.

There's a line of logic stretching from Aristotle to artificial intelligence, writes **ROBYN ARIANRHOD**, and standing near its crest is the witty and brilliant Briton Bertrand Russell.

Every man, wherever he goes, is encompassed by a cloud of comforting convictions, which move with him like flies on a summer day."

The inimitable British philosopher-mathematician Bertrand Russell, a master of pithy word-images, won the 1950 Nobel Prize for Literature for his vast body of lucid writing on science, philosophy, and social issues. And we can surely forgive his generic use of "man" in this quote, for Russell was born 150 years ago on 18 May, and he was such a supporter of women's rights that in 1907 he stood for parliament on the risky platform of votes for women.

He wasn't successful and copped extraordinary vitriol for his efforts – as he did for his advocacy of birth control, for more relaxed attitudes to sex, and for his anti-war protests. But let's get back to that statement about convictions and flies, and others in a similar vein. "People don't seem to realise that it takes time and effort and preparation to think," he noted, adding sardonically, "Statesmen are far too busy making speeches to think." And here's another classic, "Many people would sooner die than think. In fact they do." But these witticisms reflect not hubris, but his passionate interest in thinking. Not just any old thinking, either, but how to think *logically*.

Everything I say is a lie.

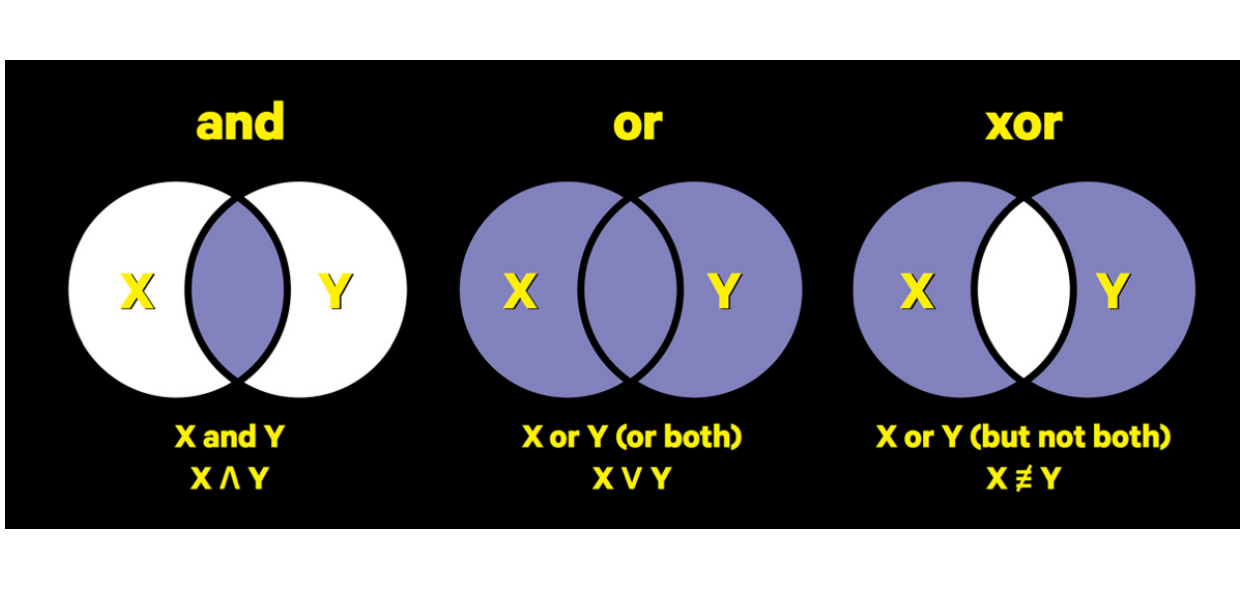
"It is undesirable," he said, putting it mildly, "to believe a proposition when there is no ground whatever for supposing it is true." Today he might have applied this to conspiracy theorists and "fake news"-evidence-deniers, but Russell applied this kind of empirical logic to social issues such as war and religion – he was an atheist, and long before Richard Dawkins and Christopher Hitchens he was writing controversial popular books such as *Why I am Not a Christian*. But whatever one's spiritual inclination, the point about debating something logically is not *belief* but logical *structure*.

Of course, we're not all Vulcans like *Star Trek*'s Spock, so logic isn't the last word either. But it's certainly helpful, so let's take a closer look at what goes into making argument, and how it leads to the rules of formal logic – especially symbolic and mathematical logic, in which Russell made major contributions.

Obvious does not equal correct

What's the framework of logic? Is it art or is it science? Aristotle set the scene more than two thousand years ago, with classic arguments such as: "All men are mortal; Socrates is a man; therefore Socrates is mortal." There are three categorical statements here; the first two are premises, from which the third follows – it's a structure now known as a syllogism. The ancient Indians and Chinese also developed logical systems of thought. But modern, symbolic logic began to arise in the mid-19th century – and it had a great deal to do with the development of symbolic algebra, which was taking shape at the same.

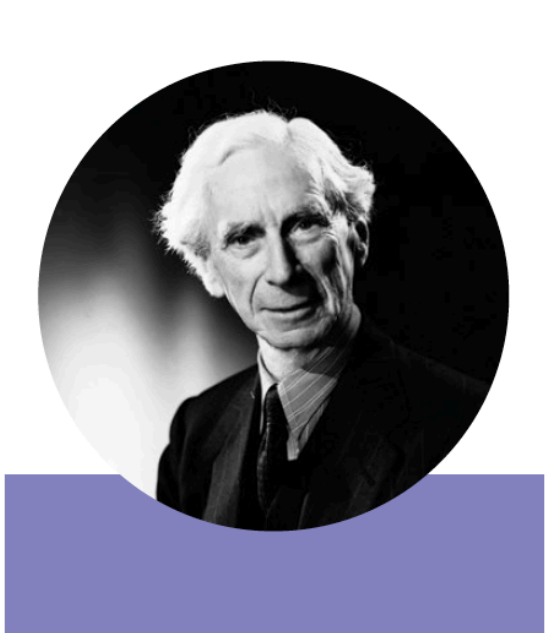
This might seem surprising, given that the ***X***s and ***Y***s we take for granted today date from the seventeenth century. But nineteenth-century symbolic algebra was designed to tease out the structure *underneath* the symbols, by identifying and naming the "commutative law," ***X* + *Y* = *Y* + *X***, and so on. The idea was that these arithmetical laws might apply not just to numbers, but to anything with the same structure – including, perhaps, logical propositions.



For instance, Russell pointed out that if a statement seems obvious, it is too easy to assume it is correct – but if you can express it symbolically, you eliminate the apparent obviousness and get down to the logic of it.

So symbolic logic is about finding a way to encode clearly and concisely the logical structure of an argument or chain of reasoning.

In his *Mysticism and Logic and other essays*, Russell expressed this idea by quoting Gottfried Leibniz, who had dreamed of a universal language for expressing the rules of reasoning in a manner similar to arithmetic. Then, said Leibniz, in any argument at all it would suffice for the disputants to "take their pens in their hands, sit down at their desks, and say 'Let us calculate!' "



> "Mathematics is capable of an artistic excellence as great as that of any music, perhaps greater … it constructs an ideal world where everything is perfect and yet true"
>
> Bertrand Russell

Credit: Hulton Deutsch / Getty

But how does one calculate reason and thought? It took two centuries to turn Leibniz's dream into reality. The person who did it was the self-educated son of a poor cobbler from regional England – yet he helped pave the way for the digital revolution. His name was George Boole, and he gave us the Boolean logic that underpins not just thought but computing, too. The way he did it was deceptively simple.

Proposition precision

We make statements all the time, generally without worrying about how two propositions might be connected logically. To take a simple example, the weather is always a good icebreaker in social interactions, such as when we say: "Nice to see the sun shining today." Boole's objective was to make precise even such simple statements as this. For a start, he said, "the sun is shining" can be true or false. If it is true, then it is certain, so it has a probability of ***1***; if it is false it has a probability of ***0***. A reply such as, "let's hope it's a nice day on Saturday or we'll have to postpone the barbecue" links two propositions, "Saturday is nice" and "the barbecue is postponed". The logical connection between them is that one will be true and the other false.

For a more high-tech illustration of this idea, the latest car safety features include such helpful prompts as beeping if you inadvertently stray across the lines marking your lane. The technology requires cameras, sophisticated software and intricate circuitry, but the ultimate aim is to tell the car's computer, "**If** the blinker is off **and** the car is crossing the line, **then** beep." As Boole pointed out, connectives such as "if… then," "and," "not," and "or" are key to the grammar of logic. But how to express it in symbolic language that a computer can read, or that humans can use to analyse arguments – that is the question!

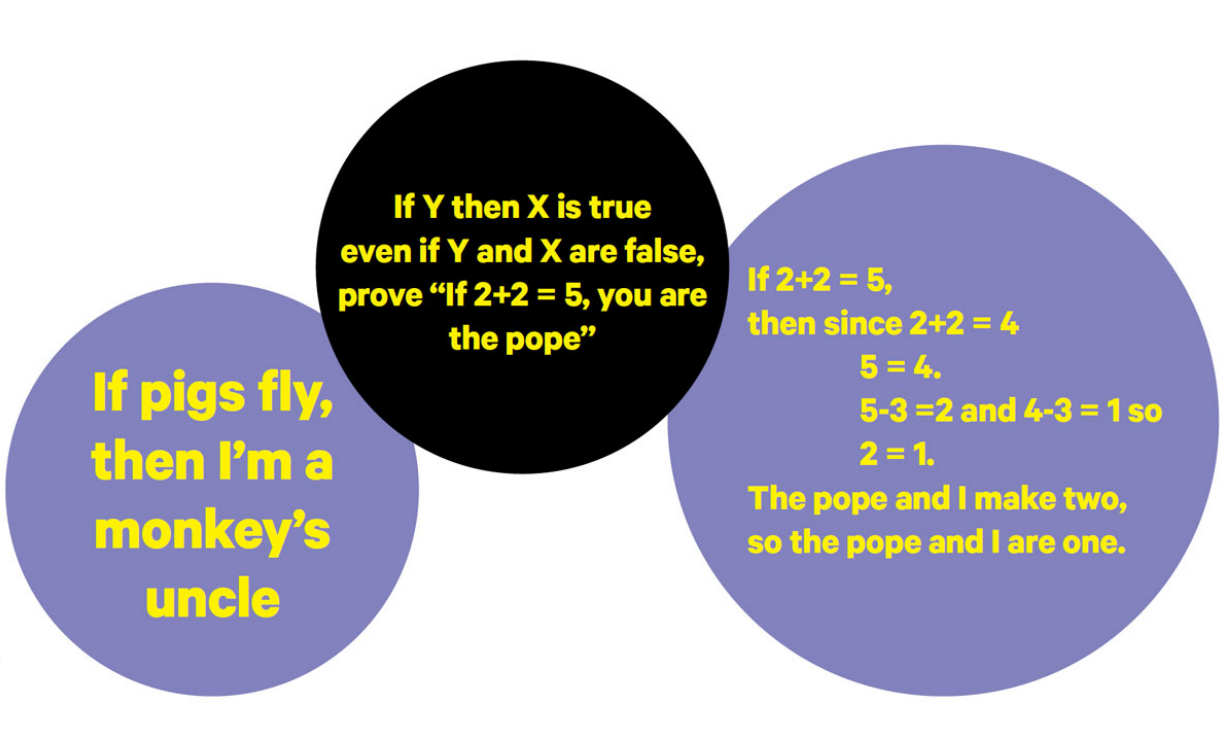
Boole's approach was to mimic the symbolism of algebra, using letters for different propositions, and the operators +, –, x and = for the logical operations ***either or***, ***not***, ***both and***, and ***is the same as***. He showed that generally these operators actually obeyed the same laws of algebra that his friend Augustus De Morgan had pioneered. After all, ***either X or Y*** is the same as ***either Y or X***, just as when ***X* + *Y* = *Y* + *X***. It's quite marvellous, really: these ancient, basic rules of arithmetic do actually encode the way the logical mind works.

Still, logic is not arithmetic. One of Boole's axioms is ***X* + *X* = *X***, because the left-hand-side says ***either X or X*** – in other words, just ***X***. In contrast to this "Boolean algebra", in ordinary algebra ***X* + *X* = *X*** only if ***X* = *0***, so you can see why later logicians chose to replace + with a symbol such as **V**. Similarly, they replaced Boole's x with **Λ**, for Boole wrote ***both X and X*** as ***X* x *X* = *X***.

The symbol **Λ** is similar to the symbol

U for the intersection of two sets, just as **V** is similar to **U** for the union of two sets. You can see the connection between sets and logic when you consider the nuances of everyday speech. Boole pointed out that when you say something like "X or Y" do you mean ***either X is true or Y is true***, or ***either X is true or Y is true or both are true***, or ***X and Y are both true***? It turns out that these different meanings can be encoded in the Venn diagrams used in set theory (see previous page).

German mathematician Gottlob Frege first put the concepts used in logic into modern set theory form in the 1880s. Russell independently developed a similar approach around 1900, and today he and Frege are considered the founders of modern logic.


If pigs fly, then I'm a monkey's uncle
If Y then X is true even if Y and X are false, prove "If 2+2 = 5, you are the pope"
If 2+2 = 5, then since 2+2 = 4 5 = 4. 5-3 =2 and 4-3 = 1 so 2 = 1. The pope and I make two, so the pope and I are one.

When logic gets complicated

Even when its operations are set out clearly like this, logic can seem bizarre rather than, well, logical. That's because logic is not concerned with what you say but simply whether or not a conclusion follows logically from its premises. It can be hard to get one's head around the idea that an argument can be valid even if some of its premises are false. Let's unpick this a little.

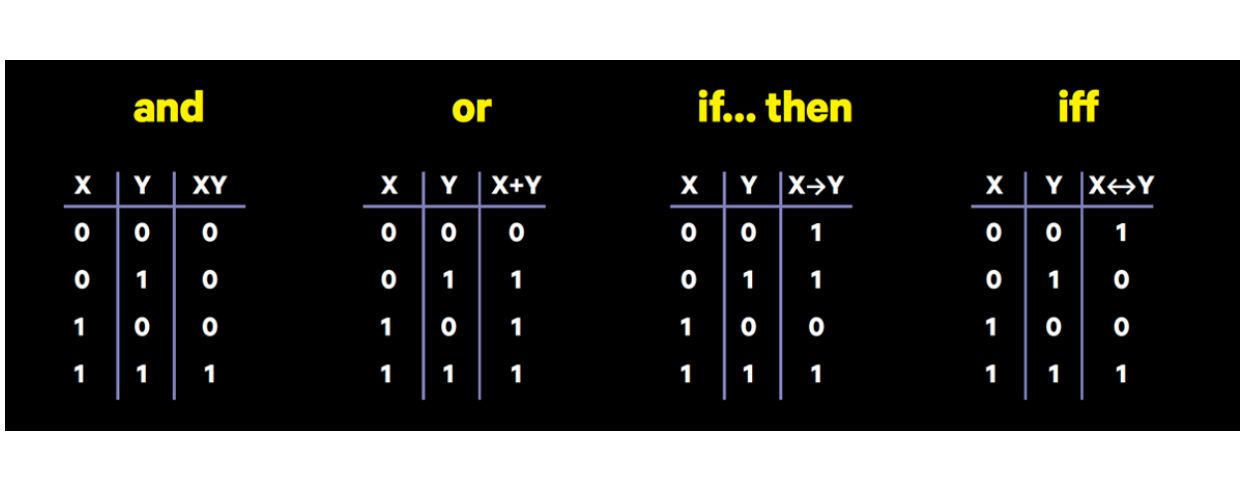
We have no trouble saying things such as "if the sun is shining we'll go ahead with the barbecue". It's a subtly different way of framing our earlier either-or barbecue proposition. Boole symbolised such statements as ***If Y is true, then X is true*** where in this case ***Y*** is the proposition "the sun is shining" and ***X*** is the proposition "the barbecue goes ahead" – and if both these things are true, then obviously so is our statement. But we're also familiar with sayings such as "If pigs can fly then I'm a monkey's uncle." This, too, is a proposition of the form ***if Y then X*** – and it is a true statement, *even though* neither ***Y*** nor ***X*** is true.

In fact, the statement is true precisely *because* neither of its propositions is true. If you're confused, take heart. There's a famous story about Russell's attempt to explain this during a seminar at Cambridge. In response, the mathematician G. H. Hardy threw out a challenge: if you can say "If Y, then X is true even if Y and X are false", then prove that "If 2+2 = 5, you are the pope" is true. Quick as a flash Russell responded along the following lines:

If 2+2 = 5, then since 2+2 also equals 4 we must have 5 = 4.

Subtract 3 from both sides of this last equation, to get 2 = 1.

But since the pope and I also make two, we have just proved that the pope and I are one.



and

X	Y	XY
0	0	0
0	1	0
1	0	0
1	1	1

or

X	Y	X+Y
0	0	0
0	1	1
1	0	1
1	1	1

if… then

X	Y	X→Y
0	0	1
0	1	1
1	0	0
1	1	1

iff

X	Y	X↔Y
0	0	1
0	1	0
1	0	0
1	1	1

It's a perfectly logical argument, which goes to show you can prove anything if you accept a false proposition as true!

More seriously, showing that ***if Y then X*** is true even if ***Y*** and ***X*** are false comes out in the rules of logic. And this is the important thing: these rules do have to be defined, just as the laws of mathematics have to be defined. The Venn diagrams are foundational, but they are not enough to say more complicated things than ***either X or Y***, ***both X and Y*** and so on. So the rules for ***if Y then X*** and other more complicated arguments need to be defined according to the "propositional calculus" that arose following Boole's work.

This "calculus" is often expressed in terms of "truth tables," where a statement such as ***if Y then X*** is evaluated depending on the truth (represented by ***1***) or falsity (represented by ***0***) of its propositions (as above). Lewis Carroll – aka Oxford mathematician Charles Dodgson, and author of *Alice's Adventures in Wonderland* – was one of the pioneers in writing out logical statements in tabular or diagrammatic form. John Venn – he of the eponymous circles – was another. But as I've implied, the language of set theory also enables the content of truth tables and other logical diagrams to be expressed purely symbolically.

Mathematical proofs obey the same logical rules of reasoning. Russell was so entranced by this connection between logic and maths that he set out to deduce, entirely from the axioms of set theory and the rules of logic, the fundamentals of mathematics itself.

The result was the remarkable three-volume *Principia Mathematica* (*PM* for short), written with his former Cambridge professor Alfred North Whitehead. Russell said he was indebted to Italian mathematician Giuseppe Peano, whom he'd met at an International Congress of Philosophy in 1900. Russell was struck by the fact that Peano "was always more precise than anyone else, and that he invariably got the better of any argument… I decided that this must be owing to his mathematical logic".

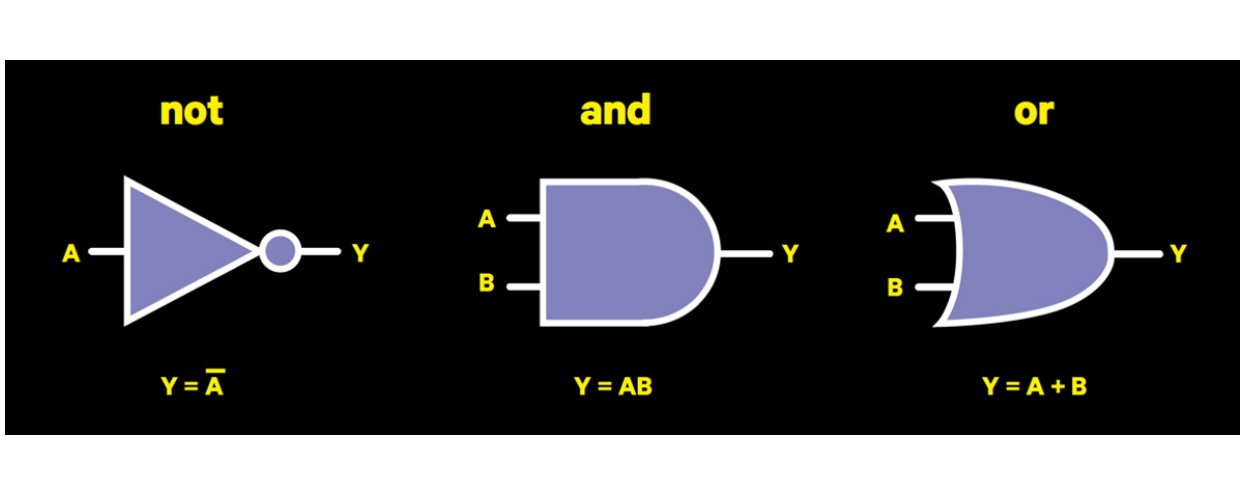
In 1931, however, the Austrian mathematician Kurt Gödel would throw a proverbial spanner right into the heart of Russell and Whitehead's attempt to deduce maths from logic. He showed that in *PM* – or any other axiomatic system about "number theory" (which studies the set of integers and their arithmetic properties) – it is simply not possible to prove *every* statement. Russell (and Spock) must have been devastated: logic was not the last word after all – at least, not when it came to deriving mathematics.

Nevertheless, in the process of writing *PM*, the first volume of which appeared in 1910, Russell had made and published many new results in mathematical logic. The most famous of these is now known as "Russell's paradox." It threw set theorists into a spin –and especially Frege, who was about to publish on the subject when he received Russell's letter about his discovery. The paradox is often presented as a story about a small village with one barber, who shaves everyone who needs

shaving – except, of course, those who shave themselves. *But who shaves the barber*? It sounds like a parlour game – but its implications are profound.

In the set theory language that Russell used, what this suggests is that contradictions arise when you bring in *self-reference*, by talking about "the set of all sets that are not members of themselves" (or the barber who shaves "all those who don't shave themselves"). For example, a forest, a set of trees, is not a tree, so it is not a member of itself. (By contrast, a catalogue of catalogues presumably lists itself, so it *is* a member of itself.)

But when you take the set of *all* sets that don't contain themselves – let's call it **ȹ** – you have a problem: where does **ȹ** itself fit in? It can't be a member of itself, because that is a contradiction: **ȹ** contains only those sets that *don't* contain themselves; similarly, if it is a member, then it shouldn't be – just as the barber belongs neither to the set of people who shave themselves nor to the set of people that don't.



The barber analogy was suggested to Russell in response to his paradox, although he felt it was too simplistic. Russell called propositions with a single verb "atomic" – Boole's "the sun is shining" is an example – so you can see that both logic and set theory have a lot to do with language. In this case, the problem was with the word all – the set of *all* sets that do not contain themselves, or the barber who shaves *all* those who do not shave themselves. Russell pointed out the similarity to the "liar paradox" connected with the semi-mythological ancient Cretan Epimenides, who supposedly said, "All Cretans are liars." If he's telling the truth, then he's a liar – and vice versa.

Russell's solution was his "theory of types". You can say, "all atomic propositions are either true or false" – but the statement itself is not an atomic proposition. It is a statement, a proposition, *about* atomic propositions, and so the set of such statements belongs to a different (higher order) degree of proposition than does the set of atomic propositions. Each time you want to say something about a set (or set of sets, or….), that something is not part of the set, so you avoid the contradictions that arise from self-reference, when sets both are and are not members of themselves.

Resolving the paradox

Russell, however, was not one to ignore evidence in order to keep to his "comforting convictions." He knew that his own resolution of the liar-barber paradox hadn't pleased all the logicians, including his famous student Ludwig Wittgenstein. His protégé George Spencer-Brown didn't like it either, and developed a startling alternative in *Laws of Form* (1969), his remarkable re-imagining of formal logic. To his relief, his mentor was delighted: Russell had always thought his theory of types was just a stopgap – and it was thanks to Russell's urging that Spencer-Brown's publisher took on *Laws of Form*.

George Boole's principles set a structure for logical thought that forms the basis of binary and complex computing. Dea / Biblioteca Ambrosiana / Getty

Spencer-Brown's "resolution" essentially said that the paradoxes of self-reference were not so bad. They're there in physics, because we're both part of the universe *and* making theories about it; and they're there in mathematics, when we solve equations such as ***x2* = *-1***, which is self-referential when you rewrite it as ***x* = *-1/x***. Yet mathematicians got around this problem by broadening the category of number to include not just positive, negative or zero numbers, but imaginary ones as well. So, said Spencer-Brown, you can get around the paradoxes of logic and set theory by broadening Boolean algebra, so that it includes not just true or false (or meaningless) statements but also "imaginary" or recursive ones.

There's a whole lot more to *Laws of Form*, but what fascinates me is the kind of creative thinking that Russell's paradox – and all the associated pioneering work in logic and symbolic algebra – inspired.

In spite of Spencer-Brown's bold take on self-reference, others *have* actually found inspiration in Russell's theory of types – for example in the software for "proof assistants". Russell and Whitehead had expressed the rules of mathematics in a purely symbolic, set-theoretic and logical way, which meant that in theory mathematical proofs could be checked mechanically. As Cornell professor Robert Constable has noted, today's proof assistants are helping mathematicians to realise *PM*'s dream.

So *PM* has had a huge impact on the development of logic and mathematics, even making its influence felt in computer science. At the time, though, it was so esoteric that the authors had to contribute to the cost of its publication. As Russell wryly noted: "We thus earned -£50 each for ten years' work. This beats *Paradise Lost*."

From Aristotle to AI

Boole, too, has been hugely influential. In 1938, an American MIT masters student, Claude Shannon, published a thesis showing how Boolean logic could be represented as electronic circuits. First there are switches (or voltage regulators), where ***on*** (or "high" voltage) represents ***1***, and ***off*** (or "low" voltage) represents ***0***. Then you can connect two switches in series to represent ***X and Y are both true***, while connecting them in parallel represents ***X or Y is true*** – with more complicated circuits representing more complicated propositions. The electronic digital revolution was on its way; all it needed was for someone to invent a computer!

One of the first steps towards this goal was the thought-experiment that became known as a "Turing machine". While Shannon cited Boole, the British mathematician Alan Turing took his lead from Russell (and Gödel, who'd taken *his* lead from *PM*). Turing is famous for his use of logic in breaking the German Enigma code during the Second World War – and for the tragedy of his 1954 suicide, apparently due to the homophobia that saw him arrested in 1952. He was born in June 1912, so we can celebrate his 110th birthday along with Russell's 150th.

While Gödel showed that not all theorems are *provable*, in 1936 Turing showed that not all mathematical quantities are *computable*. He defined real numbers as "computable" if there is an algorithm that generates them. For instance, you might think that **π** is not computable, but it can be generated by algorithms such as

$$^{\pi}/4 = 1 - 1/3 + 1/5 - 1/7 + 1/9 - \ldots$$

which comes from the series for **tan-*1x***. What Turing discovered, though, was that if you tried to test for computability in advance, you ended up with yet another analogue of the liar or barber paradox! In other words, he showed that it was not possible, even for a computer, to compute everything. It had to do with the problem of self-reference again – just as we had with Russell's paradox and sets containing themselves, and with Gödel's axiomatic systems, such as *PM*, trying to prove themselves complete.

Credit: Pictures from history / Getty

It's intriguing, because the problem of self-reference also comes up in the most mysterious phenomenon of all, human consciousness, which not only perceives the outside world but itself, too. Turing was fascinated by the idea that our minds work like computers – not in their hardware but in their reliance on algorithms.

In 1950 he kick-started the field of artificial intelligence (AI) when he proposed his famous take on the Victorian "imitation game," now known as the "Turing test". The idea was for the player to pose various questions to an unseen human and a computer, who would reply in writing so as not to give the game away. If the player couldn't tell which was which, then the computer was deemed to be "thinking".

Machines are getting "smarter" all the time, although none has passed the Turing test. But the lead scientist at Amazon's Alexa program, Rohit Prasad, believes that AI is now so advanced and so ubiquitous in our smart technology that the Turing test is no longer a useful benchmark – and he's not alone.

Away from high-tech labs, philosophers and mathematicians, too, are debating the relationship between intelligence and machines (or algorithms) – and the reach of Gödel's and Turing's results on provability, computability, and AI.

Perhaps the human mind can do more than a computer ever could – and perhaps human creativity, with its insight and its human error, makes Gödel's theorem irrelevant: we make mistakes anyway, so does the "incompleteness" of maths really matter? Is human creativity more than computability, or a behaviourist analysis such as the Turing test?

And Russell's own philosophical work on a theory of mind and matter called "neutral monism" has been having renewed impact on the study of consciousness, thanks initially to Australian philosopher David Chalmers.

So on Russell's 150th birthday I think it's good to pause amid the excitement of ever-advancing technology, and remember the philosophers and mathematicians, past and present, who have not only helped make it all possible, but who have also stopped to ask questions about how we think and who we are.

ROBYN ARIANRHOD is an affiliate in mathematics at Monash. Her last story, on algebra, appeared in Issue 92.

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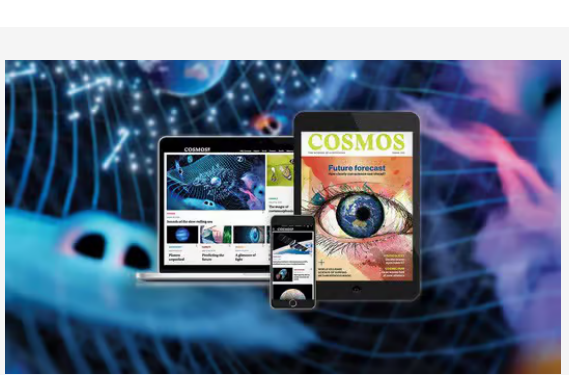

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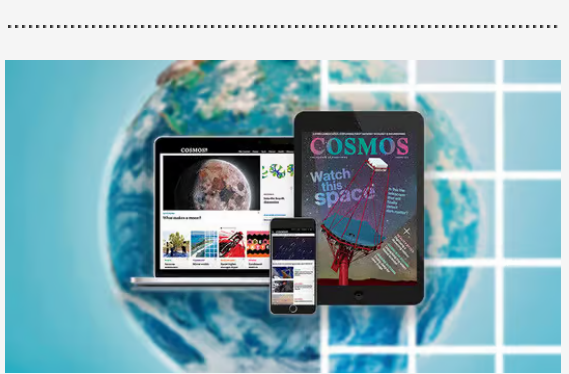

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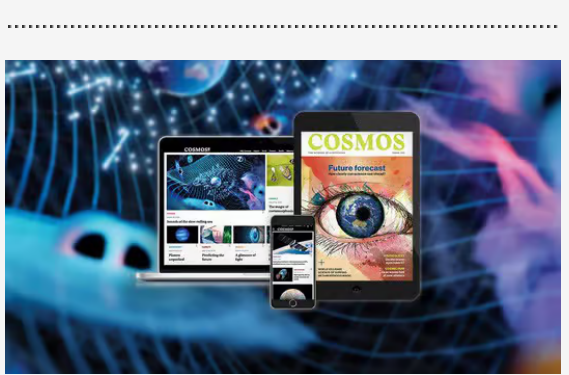

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