

# THE LEGEND OF MAXWELL'S EQUATIONS

If you've studied high school physics you've heard the name Maxwell – but even if you haven't, **Robyn Arianrhod** shows why his famous equations are at the heart of modern life. She also highlights the mathematical elegance of these equations and shares the thrilling tales behind their discovery.

**M**axwell's equations of electromagnetism are legendary. Every time we turn on a light, every time we use a computer, a mobile phone, and just about all our other electrical tech, these famous equations are somewhere there at the heart of it. In their modern vector form, they are also perhaps the most beautiful equations in all of physics.

But there's another legend surrounding these elegant equations: the claim that Maxwell was not their true author. It first surfaced as a thrilling tale of a long-overlooked outsider besting one of the giants of physics. We'll meet him shortly, along with James Clerk Maxwell himself – and we'll see why much of the thrill in this legend is illusory. First, though, let's see what Maxwell's equations actually do.

**Elegant equations**

The first equation in the box tells physicists how the electric field produced by an electric charge spreads out through space, while the second one does the same for the magnetic field around a magnet.

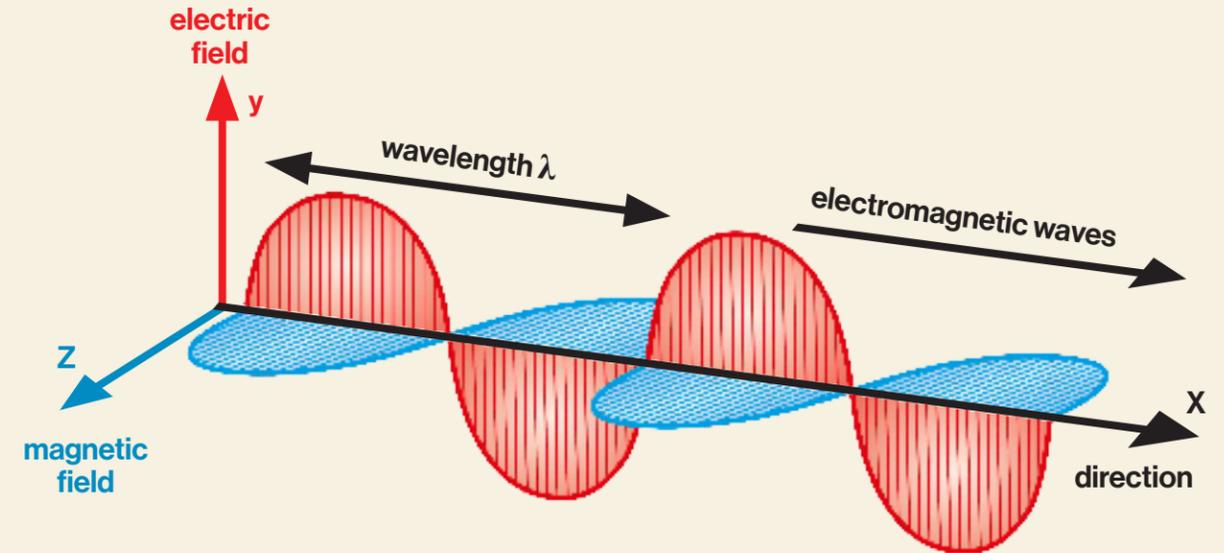
Things become more interesting with the third equation. It shows that a changing magnetic field produces a new *electric* field. For instance, moving a magnet through a loop of wire can cause an electric current to flow in the

Maxwell's equations

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 4\pi\rho \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{B} &= \frac{\partial \mathbf{E}}{\partial t} + 4\pi\mathbf{J} \end{aligned}$$

Maxwell's equations of electromagnetism (with units chosen so the electrical and magnetic constants are set to 1). The beauty lies in the visual assonance here, and the symbolic interweaving of **E** and **B** that illustrates the entwined physical electric (**E**) and magnetic (**B**) fields that give us light and enable our wireless technology.

In more complex cases, **E** is incorporated into a broader term **D**, the electric displacement, while **B** is related to the magnetic field strength **H**. Maxwell and Heaviside used these more general definitions in their equations, but for simplicity this story focuses on **E** and **B**.



wire – as if conjured out of thin air! This is the basis of technologies such as battery and EV chargers, and electric generators that power the grid: using wind or waterpower, or steam produced from burning coal, huge coils are turned through magnetic fields to produce most of the electricity that runs our lives. (The photovoltaic cells you may have on your rooftop, by contrast, use light energy to produce a current, courtesy of quantum mechanics as well as Maxwell's equations.)

As for the fourth equation, it shows that this remarkable connection between electricity and magnetism is two-way: a changing electric field can move a magnet – with the kind of rotational deflection that is now the basis of the electric motors you find in everyday items such as blenders, electric toothbrushes, automatic garage doors, electric drills and fans... you get the picture.

Many technologies use both these last 2 equations, of course. In induction cooktops, for example, an alternating current through a coil underneath the hotplate produces an oscillating magnetic field in the bottom of a suitable saucepan; in turn, this changing magnetic field induces

“eddy” currents in the saucepan. Roughly speaking, heat is produced when the currents' electrons collide with the metal's atoms.

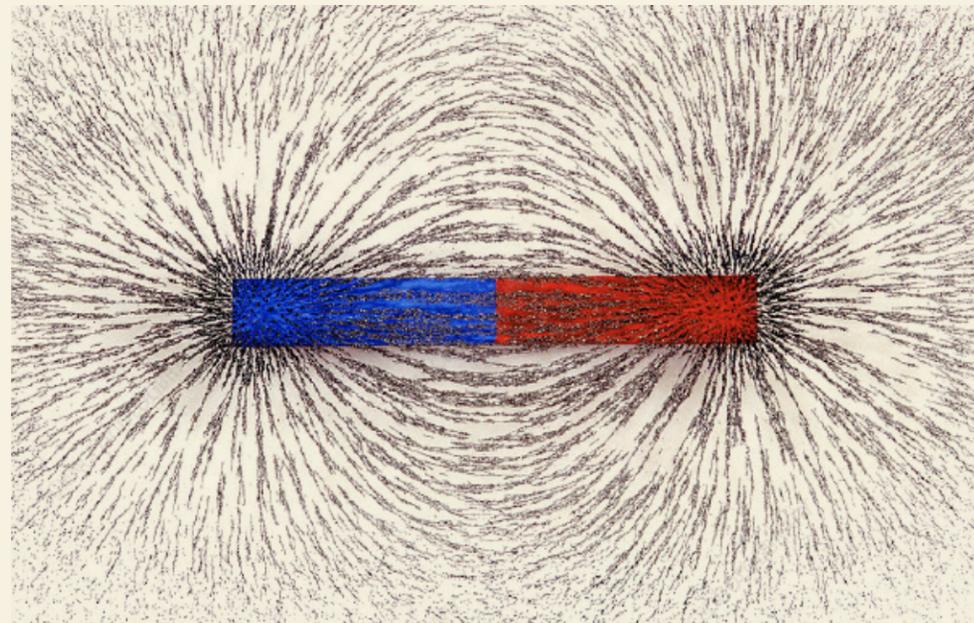
**Stepping into the lab**

Maxwell built these laws from experimental discoveries made by Charles de Coulomb, Carl Gauss, Hans Ørsted, André-Marie Ampère, and Michael Faraday.

And in truth, says Monash University theoretical physicist Michael Morgan, “These experiments would likely have led to some of our electrical technology even without Maxwell – although having the equations certainly makes technological innovation easier.”

But, Morgan adds, it's a different picture when it comes to some of our most remarkable technologies, wireless and optical telecommunications. (We're chatting via Zoom, a case in point.) For Maxwell did much more than work out the maths to describe known experimental results. He discovered that putting these 4 equations together, you get the equations for electromagnetic waves – the things that give us light and optical technology, radio and television, Wi-Fi, MRI and other medical imaging, optical and radio astronomy, and much more.

“Perhaps someone would have serendipitously discovered electromagnetic radiation and all its properties,” Morgan muses. “But having a theory that makes predictions about the existence and nature of such radiation, you're much more likely to look for it” – as Heinrich Hertz famously did, creating and detecting electromagnetic waves in 1886.



Here you can see iron filings responding to a bar magnet's magnetic field. The field lines diverge from one pole to the other, so there is zero total divergence (hence the 0 in the second equation).

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Morgan's Monash colleague Lincoln Turner, an experimental physicist, works closely with Maxwell's equations. "I can't think of a physicist who doesn't, directly or indirectly," he tells me, for Maxwell's equations explain the entire electromagnetic spectrum.

"Since this includes radio waves, microwaves, infrared, visible light, UV, X-rays and gamma rays," Turner says, "it covers almost all of how we observe the universe, and a very large part of how we interact with it."

For example, while building cold atom magnetometers recently, Turner needed to measure how much laser light is lost through reflections from a glass vacuum chamber – "just as you can see your reflection when you look through a clear glass window," he says, noting that the calculations are "a direct application of Maxwell's equations."

And just last week, Turner adds, "I was working on polarisation of the magnetic component of microwave fields, that we use to control ultra-cold atoms and make atomic clocks out of them. Maxwell's equations give us a complete picture of what the microwaves are doing."

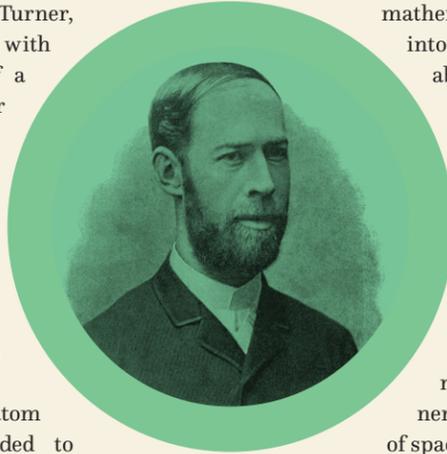
But how did Maxwell do it? It's time to backtrack and see how his discoveries unfolded – and to find out just who did write "Maxwell's equations".

### Who did what?

The work of the experimental pioneers, from Coulomb to Faraday, spanned the half century between 1785 and 1831, the year Maxwell was born. Yet 25 years later, when Maxwell was a young Cambridge graduate, no one had been able to figure out just *how* these newly observed electric and magnetic forces travelled and interacted.

No one, that is, except Faraday. He is one of Turner's experimentalist heroes, partly because of the "profoundly creative insight" that envisaged the electromagnetic field. Faraday believed that *something* must mediate the forces emanating from electric and magnetic sources, akin to the way a breeze rippling through a field of wheat bends each stalk in turn. For no one in their right mind, he thought, could believe that these forces acted instantaneously, at-a-distance – leaping instantly from one magnet or charge to another, or from a magnet to a loop of wire.

Trouble was, most physicists *did* think in this way, and they refused to take seriously Faraday's "field" idea. But Maxwell loved it. The self-taught Faraday had lacked the



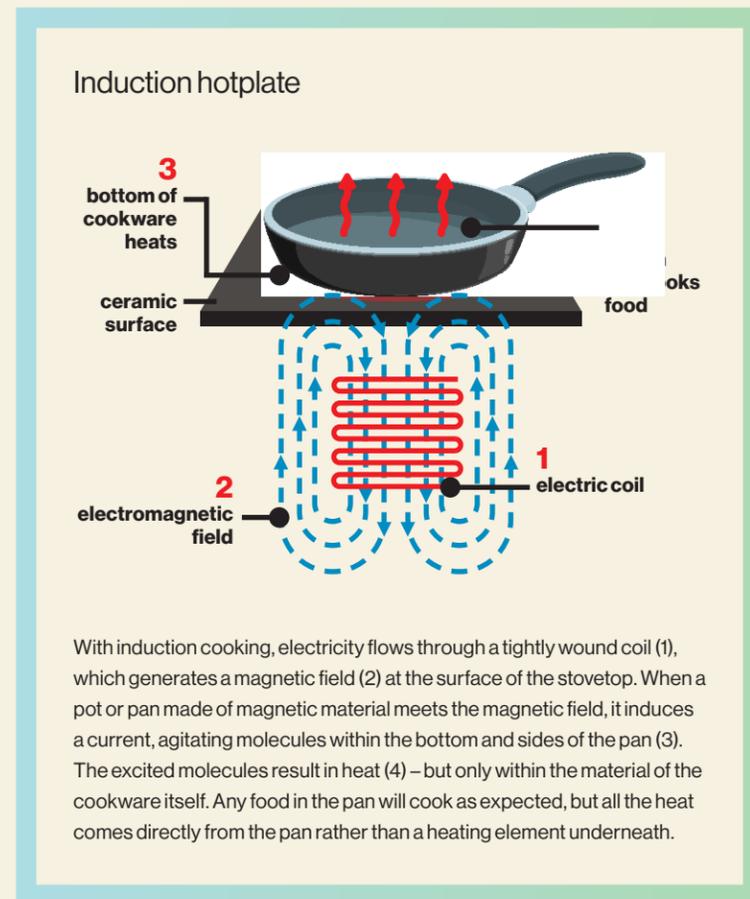
German physicist Heinrich Rudolf Hertz (1857–1894) was the first to create and detect electromagnetic waves.

mathematical knowledge to translate his idea into predictive equations, so Maxwell set about trying to find a way to express all the known facts about electricity and magnetism in terms of fields.

The first thing he did was to identify electric and magnetic forces with vectors: they have both magnitude and direction. Isaac Newton was the first to define forces in terms of these 2 properties, and what it meant mathematically was that for each force, you needed an equation for each of its 3 components – one in each of the x, y and z directions of space. Maxwell defined Faraday's electric and magnetic fields as sets of vectors at each point in space.

The second thing he did was to choose differential calculus to describe the way these vectors changed through space and time.

This new vector field language – plus an inspired addition of his own (called the electric displacement) – enabled Maxwell to formally unite, for the first time, all that was known about electricity and magnetism. And it is this language that enabled him to deduce the "wave



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### WAVELENGTHS (Metres)



### VISIBLE SPECTRUM



### FREQUENCY (Hz)



equations," which predicted that these forces do not act remotely: their effects propagate as waves through the electromagnetic field.

Maxwell quickly realised that his mathematical waves had the same shape and speed as light waves. Experiments had shown that light travelled as a wave, although no one knew what, exactly, was waving.

So, Maxwell made his bold prediction, that "light itself (including radiant heat, and other radiations if any)," is *electromagnetic*, a wave of rippling electric and magnetic fields. Today, we know that Maxwell's "other radiation" includes all the kinds that Turner alluded to, from Hertz's radio waves to gamma radiation.

### Enter Heaviside

Maxwell first published his extraordinary theory in 1865. And now we come to our eccentric outsider, the acerbically witty telegrapher and self-taught physicist Oliver Heaviside.

At the time of writing, Wikipedia states that "Heaviside employed vector calculus to synthesize Maxwell's over 20 equations into the four recognizable ones which modern physicists use." Another entry says that Heaviside "independently" developed vector calculus, and used "modern vector terminology to reduce 12 of the original 20 equations" down to the 4 "Maxwell's

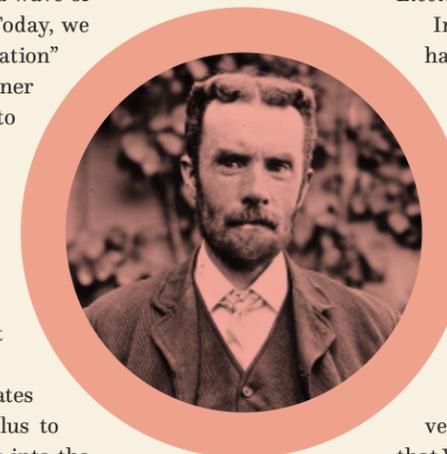
equations". Such claims are widely repeated, but they are only partly true.

Heaviside is a breath of fresh air who did, indeed, contribute significantly to making Maxwell's equations more modern and beautiful. But he was clear about his debt to the man he called "the heaven-sent Maxwell". For he had read not Maxwell's original, oft-cited 1865 paper – which did present 20 equations of electromagnetism – but his 1873 masterpiece, *Treatise on Electricity and Magnetism*.

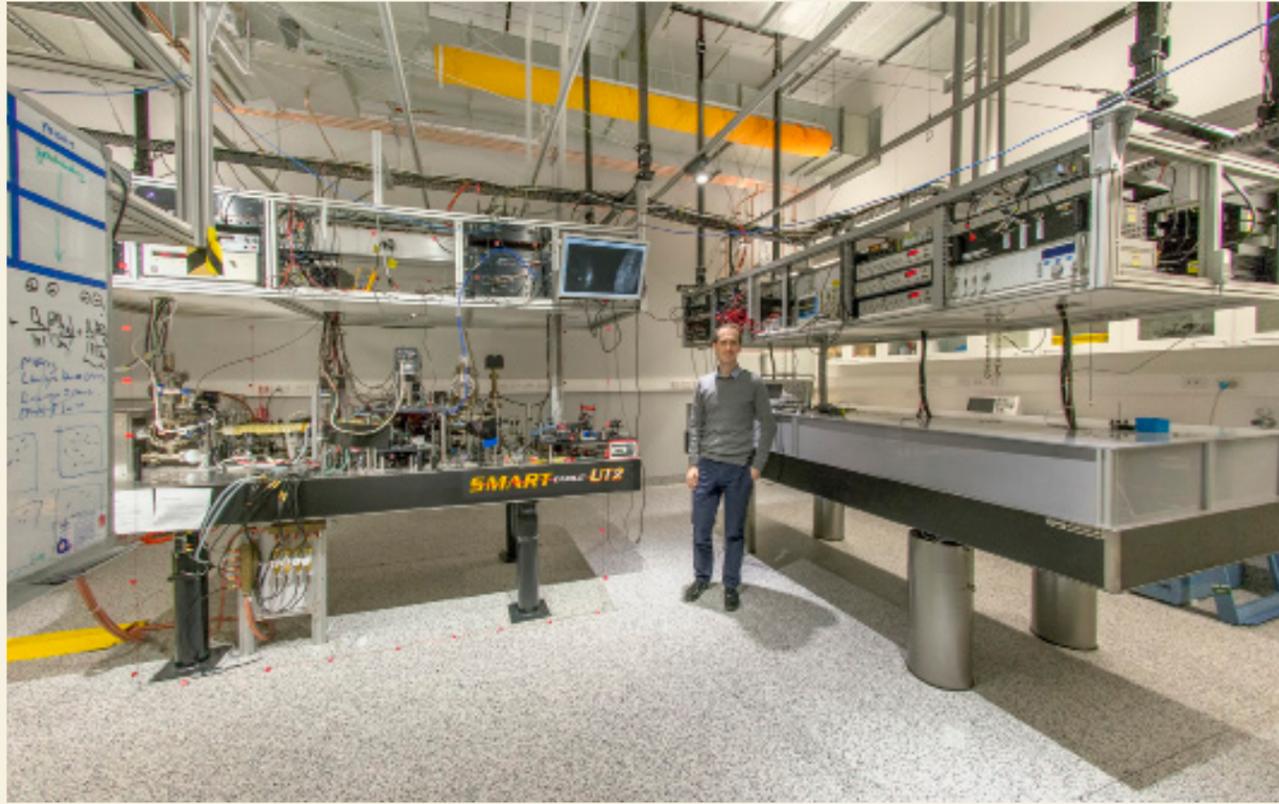
In the years between 1865 and 1873, Maxwell had learned more about the new vector calculus created by William Rowan Hamilton. Ever since Newton, people had known that forces had 3 directional components, but Hamilton figured out how these separate components behaved as a whole – a whole for which he coined the name "vector."

In his *Treatise*, it was Maxwell himself who reduced those 12 component equations to 4 whole-vector ones. In fact, it is Maxwell who inaugurated the vector calculus terms "curl" and "divergence" that Wikipedia implicitly ascribes to Heaviside.

In the modern set of equations, the  $\nabla \cdot$  symbols denote the divergence, while  $\nabla \times$  denotes the curl. These names and symbols embody the physical reality of the diverging forces and twisting, "curling" behaviour that



Oliver Heaviside, circa 1900.



experimenters had observed – as when iron filings line up in circles around a current-carrying wire, for instance, and in the torque that now drives electric motors.

Maxwell was the first major physicist to use this new vector language, because he felt that a single suggestive equation made it much easier to see the physics than if you had 3 separate component equations for each experimental result.

When Heaviside read Maxwell's *Treatise* he was awestruck, describing it as “great, greater, and greatest”. It is from this book that he discovered the existence of Hamilton's vector calculus, which was still so controversial that debates about its utility would rage for nearly 2 decades after Maxwell's death in 1879.

Vector symbolism itself was still being developed at that time, and Maxwell had chosen to represent his vectors with Gothic letters, such as  $\mathfrak{B}$  and  $\mathfrak{C}$ . We can thank Heaviside for changing these to the more user-friendly  $B$  and  $E$ . But it was Heaviside's American contemporary Josiah Willard Gibbs who gave us the dot and cross in the divergence and curl operators, which also contribute to the beauty of the modern vector form of Maxwell's equations.

Still, in unpacking the authorship question, it's important to note that the content of a single vector equation is exactly the same as the content

**Lincoln Turner, an experimental physicist, in his lab at Monash University.**



**Monash University theoretical physicist, Michael Morgan.**

of its 3 component equations. In fact, this is why critics such as Lord Kelvin had seen no value in the new vector calculus – Hamilton's or Heaviside's and Gibbs's.

### Future potential

Heaviside did bring out the beautiful symmetry between the electric and magnetic fields, by bringing them to the fore. Maxwell's 4 mathematically equivalent equations had been written in terms of both the  $E$  and  $B$  fields and a concept called the “potentials.” There are 2 electromagnetic potentials.

First, an electric “scalar” one, often denoted by  $\phi$ ; this is related to the voltage, and the potential energy of a charge. Potentials had been used in physics, especially gravitational physics, for a century before Maxwell wrote his equations.

Second, a more mysterious magnetic “vector” potential field,  $A$ . Maxwell explained that this arises from the physics (Gauss's laws) via mathematics (Stokes's theorem) – but the point here is that he was concerned with mathematical representations of experimental results.

By contrast, Heaviside, the telegraphic engineer, was concerned with practical, physical measurements (in “wire-ful” rather than “wire-less” tech, quips Morgan). That's why, in the 1880s, he rewrote Maxwell's vector equations

solely in terms of  $E$  and  $B$ . For it is these fields that are directly measurable, in the sense that the measured forces acting on particles or magnetic dipoles moving in an electromagnetic field can be found quite directly from  $E$  and  $B$ .

Heaviside's formulation has been incredibly useful. Mathematically speaking, though, his “rewrite” of Maxwell's equations is not quite as amazing as the legend suggests. Using vector maths, it follows easily from Maxwell's own version, as Heaviside was the first to acknowledge.

What's more, today Maxwell's potentials – which Heaviside had gleefully “murdered,” as he put it – are alive and well, and not just in electromagnetism, but in relativity and quantum mechanics, too. That's because the potentials can simplify calculations.

In fact, Turner notes, “In quantum mechanics we use  $\phi$  and  $A$  – the vector potential  $A$  appears naturally when quantising the momentum. If we do need  $E$  and  $B$ , we can calculate them from the potentials.” Morgan agrees: “It's damn hard to form a local theory in  $E$  and  $B$ , but it's natural in  $\phi$  and  $A$ .”

## Maxwell's equations laid foundations for the relativity and quantum revolutions.

Things had taken an interesting turn in this direction back in 1959, when Yakir Aharonov and David Bohm, then at the University of Bristol, made a radical suggestion: in the quantum mechanical domain, both these potentials might, after all, represent something more fundamentally physical than Heaviside's fields. This idea got a huge boost in the 1980s, when the hypothetical “Aharonov–Bohm effect” was unequivocally demonstrated in what Morgan describes as “a beautiful experiment,” by Akira Tonomura and his colleagues.

The Aharonov–Bohm effect is an observable phase shift in the wavefunction of a charged particle in regions where the magnetic field is zero but, surprisingly, the vector potential is not. Suddenly, this strange kind of potential had a physical effect, not just a mathematical one.

Not everyone agrees that the potentials are more fundamental than the electric and magnetic fields. But that's a philosophical question, says Morgan, whereas physicists use whatever works for them. Turner agrees. But having measured phase shifts in wavefunctions himself (using atom interferometers), he can't help but feel that “they make a very compelling case for  $A$  being more ‘real’ than  $B$ .”

### Lasting legacies

Heaviside made many significant discoveries in electromagnetics. He also spearheaded the formulation of modern, post-Hamilton vector calculus, and he did help to make Maxwell's equations more beautiful. But as he explicitly noted, they *are* Maxwell's equations.

They enabled Maxwell to unify electricity, magnetism and light into a single, profound and elegant theory, which also laid physical and mathematical foundations for the relativity and quantum revolutions that followed it. “Imagine Maxwell's feelings,” Einstein enthused, “at that thrilling moment” when his theory all came together. It's a remarkable legacy.

Sadly, Maxwell never got to see his theory confirmed. But we all use its fruits every day.  $\odot$

**ROBYN ARIANRHOD** is a writer and historian. Her latest book is *Vector: A Surprising Story of Space, Time, and Mathematical Transformation*. She is an affiliate with Monash University's School of Mathematics.

### The Aharonov–Bohm effect

Electrons from 2 slits pass by a solenoid (an electromagnet) on its upper and lower side. The electrons form interference patterns on the screen. Without a magnetic field in the solenoid, the interference pattern is like the yellow palette. With a magnetic field inside the solenoid but not outside, the electrons form the interference pattern shown on the red palette. In quantum mechanics, the result is interpreted as being directly related to the vector potential which causes the shift of the interference pattern.

AN INTERPRETATION FOR AHARONOV-BOHMEFFECT WITH CLASSICAL ELECTROMAGNETIC THEORY, GAOBIAO XIAO, SHANGHAI JIAO TONG UNIVERSITY, DOI:10.48550/ARXIV.2201.12292, CC BY 4.0